Extracting the Cliques from a Neighbourhood System

Rupert Paget and Dennis Longstaff

Abstract—**A method is proposed for obtaining the local clique set from a neighbourhood system. The Markov random field model, which is used extensively in image processing, is defined with respect to a neighbourhood system. The mathematical interpretation of the model is defined with respect to the corresponding clique set. We present a systematic method for extracting the complete local clique set from any neighbourhood system on which a Markov random field maybe defined.**

Keywords—**Markov random field, Neighbourhood systems, Cliques**

I. Introduction

Markov Random Field (MRF) models are used in image restoration [5], region segmentation [4] and texture analysis [2]. However the preferred method of analysis in these applications is to use the equivalent Gibbs Random Field model. To obtain this Gibbs model it is first necessary to extract the local clique set from the neighbourhood system defined by the MRF model. This is a complex combinational problem for large neighbourhood systems for which we propose a method to systematically extract the local clique set from any neighbourhood system. Although in practice mostly small neighbourhood systems are used, which may not necessarily benefit from this systematic method of extraction, we found it invaluable in our experiments when we were able to use large neighbourhood systems for nonparametric MRFs.

The property of an MRF is that given a point on a lattice, the probability of that point being set to any particular value is conditional upon the values of its "neighbouring" points defined by a neighbourhood system. In other words, the MRF is characterised by a local conditional probability function defined with respect to a neighbourhood system. An equivalent Gibbs Random Field defines its probability function over a set of local cliques which are subsets of the neighbouring points [1].

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II. Neighbourhoods and their Cliques

A comprehensive examination of Markov random fields is given by [3]. In this section a brief overview of the MRF theory is presented in order to give the necessary background on neighbourhoods and their respective cliques.

Denote a set of sites on a lattice by S, and the neighbourhood system over S as $\mathcal{G} = \{\mathcal{G}_s, s \in S\}$, where $\mathcal{G}_s$ is the set of "neighbours" for s such that $\mathcal{G}_s \subset S, s \notin \mathcal{G}_s$. Given the random variable X_s at site s with value x_s, the local conditional probability function of a MRF with respect to the neighbourhood system $\mathcal{G}$ is defined by the Hammersley and Clifford theorem [1] as,

$$P(X_s = x_s | X_r = x_r, r \in \mathcal{G}_s) = \frac{1}{Z_s} \exp\left\{ - \sum_{C \in \mathcal{C}_s} V_C(\mathbf{x}) \right\}, \quad (1)$$

where Z_s is a constant and V_C is a potential function defined on the clique C. The summation is over all cliques in the local clique set $\mathcal{C}_s$. The variable $\mathbf{x}$ is the set of values $\{x_s, s \in S\}$.

The Hammersley and Clifford theorem implicitly requires the neighbourhood system to adhere to the criterion that $s \in \mathcal{G}_r \Leftrightarrow r \in \mathcal{G}_s$. This implies that neighbourhoods must be symmetrical if the MRF is homogeneous. Three different neighbourhoods are shown in Figs. 1(a)–(c) which are defined by,

$$\mathcal{G}_s^d = \{r \in S : 0 < |s - r|^2 \le d\} \; \forall \, r, s \in S, \quad (2)$$

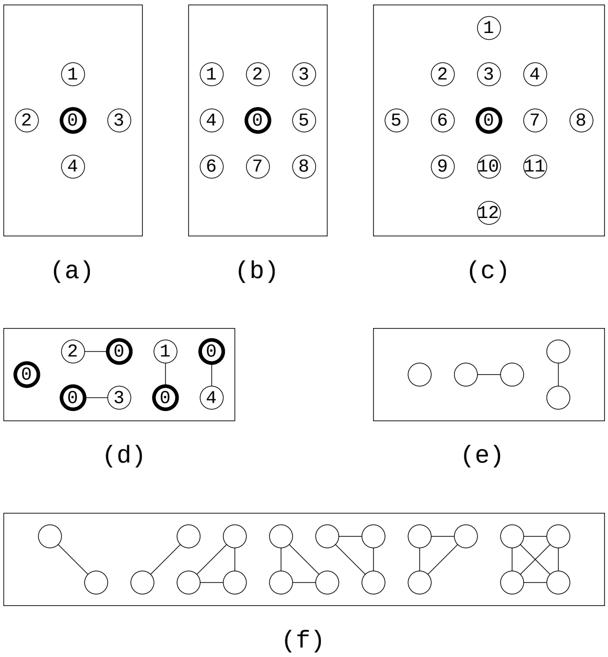

Fig. 1. Neighbourhoods and cliques: (a) The nearest-neighbour neighbourhood; (b) second-order neighbourhood; (c) fourth-order neighbourhood; (d) local clique set for nearest-neighbour neighbourhood; (e) clique types for nearest-neighbour neighbourhood; (f) additional clique types for second-order neighbourhood.

as given by [3] where d specifies the order of the neighbourhood.

Given a neighbourhood system $\mathcal{G}$, a clique is a set $C \subseteq S$ such that $s, r \in C, s \neq r$, implies $s \in \mathcal{G}_r$. That is, every pair of distinct sites in a clique are neighbours. The single site subset is also a clique. The local clique set for the site s is defined as $\mathcal{C}_s = \{C \subseteq S : s \in C\}$.

The local clique set for the first-order neighbourhood $\mathcal{G}_s^1$, Fig. 1(a), is shown in Fig. 1(d). This local clique set has three different clique types which are shown in Fig. 1(e). The local clique set for the second-order neighbourhood, Fig. 1(b), contains the clique types shown in Figs. 1(e) and (f).

III. Extraction of the Local Clique Set

The proposed method for extracting the local clique set from a MRF neighbourhood system is based on graphing a tree structure. The root of the tree represents a single site. The branches at the first level represent all the pairwise connections to the sites in the neighbourhood. Further branches at the higher levels represent high order connections that form more complex cliques.

Given a set of sites S, let $n(r)$ denote the node number of the site $r \in S$ with respect to the neighbourhood $\mathcal{G}_s$ where $r \in \mathcal{G}_s$ $s \in S$. Figs. 1(a)–(c) show the node numbers for neighbourhoods $\mathcal{G}_s^1$, $\mathcal{G}_s^2$ and $\mathcal{G}_s^4$ respectively. The node numbers used in clique trees refer directly to the sites in the respective neighbourhoods.

A. Method 1: Growing the Clique Tree

Follow the steps outlined in Fig. 2 as to how to graph the clique tree.

Method 1 *Graphing the Clique Tree*
Input:
$S = \{s, r, t, \ldots\} \leftarrow$ set of sites on a lattice
$\mathcal{G}_s \leftarrow$ neighbourhood for site $s \in S$
$n(r) \leftarrow$ node numbers for sites $r \in \mathcal{G}_s$
begin
1. Place the node $n(s) = 0$ $s \in S$ at level 1. This is the root node of the tree.
2. Place the nodes $n(r)$ $r \in \mathcal{G}_s$ at level 2.
3. Link the root node $n(s) = 0$ with an arrow to each node $n(r)$ $r \in \mathcal{G}_s$ at level 2.
4. Let $m = 1$.
5. **While** nodes exist at level $m+1$ **do**
5.1 Increment m
5.2 **For each** $n(r)$ at level m **do**
5.2.1 Let $n(s)$ be the node at level $m-1$ that directly links to the node $n(r)$ at level m.
5.2.2 Place the nodes $n(t)$ $t \in S$ at level $m+1$ which adhere to the following criteria:
- An arrow directly links the node $n(s)$ at level $m-1$ with the node $n(t)$ at level m
- $t \in \mathcal{G}_r$
- $n(t) > n(r)$

5.2.3 Link the node $n(r)$ at level m with an arrow to each node $n(t)$ recently placed at level $m+1$.
end

Fig. 2. Method 1: Graphing the Clique Tree

B. Method 2: Reading Cliques from the Tree

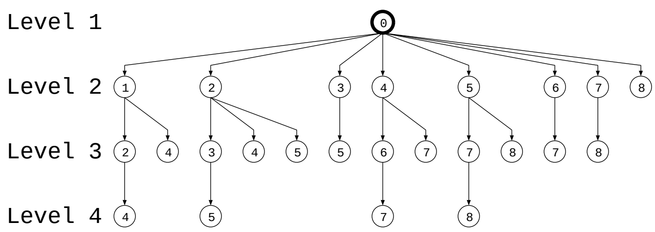

Fig. 3. Clique tree for the neighbourhood shown in Fig. 1(b).

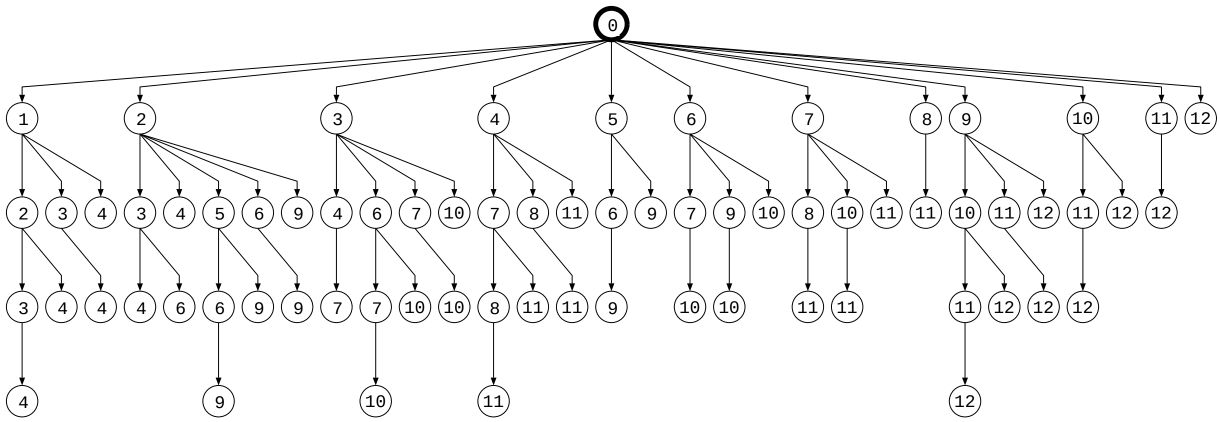

Fig. 4. Clique tree for the neighbourhood shown in Fig. 1(c).

A clique in the tree is represented as any tree transversal following the arrows from one level to the next beginning at the root node (level 1). The single site clique is represented as the single node $n(s) = 0$ at level 1. The pairwise cliques are represented as the node $n(s) = 0$ plus any other node $n(r)$ at level 2. In Fig. 3 an example of a three site clique is represented by the nodes $\{0, 2, 4\}$.

The complete set of cliques that can possibly be read from the clique tree in Fig. 3 is the local clique set for the neighbourhood shown in Fig. 1(b). The clique tree of Fig. 4 represents the local clique set for the neighbourhood shown in Fig. 1(c).

IV. Clique Tree Theorems

The following theorems prove that the local clique set of a neighbourhood is completely represented by its respective clique tree.

Theorem 1: A set of nodes derived from Methods 1 and 2 is a clique

Proof. By the construction of the clique tree, every node on the tree is contained within the neighbourhood of all other nodes on the tree that can transverse to it by following the arrows.

Theorem 2: Each clique represented by the clique tree is unique

Proof. In growing the clique tree via Method 1, a node $n(s)$ at level m only links to nodes $n(r)$ at level $m + 1$ for which $n(r) > n(s)$. This means that the nodes $\{n(s) = 0, n(r), n(t), \ldots\}$, which can be read from the clique tree via Method 2, are monotonic increasing in node number. Since the nodes are ordered, no permutations of the same set of nodes can be read from the clique tree via Method 2. Therefore each different clique read via Method 2 is unique.

Theorem 3: Every local clique is included in the tree

Proof. Consider any local clique $\{r, \ldots, s\}$ for the site $s \in S$. The clique can be rearrange into a set of monotonic increasing node numbers given by the neighbourhood $\mathcal{G}_s$ such that,

$$\{t, s \ldots, r\} \Rightarrow \{n(s) = 0, n(r), \ldots, n(t)\}. \quad (3)$$

The set of nodes $\{n(s), n(r), \ldots, n(t)\}$ cannot represent a local clique without the first node $n(s) = 0$. The next node $n(r)$ must be contained in the neighbourhood $\mathcal{G}_s$ and is therefore represented at level 2 on the clique tree. Continuing along the list, the next node must be a neighbour to each of the previous nodes. Because of the criteria stated at step 5.2.2 in Method 1, this node exists on the clique tree at level 3 and is linked by an arrow from the node $n(r)$ at level 2. By considering each node from a local clique in a monotonic increasing order, it is clear that by the structure of the clique tree, the local clique must be included in the clique tree.

Theorem 4: For each clique type with n nodes, $\exists\ n$ local cliques

Proof. A local clique of the neighbourhood $\mathcal{G}_s$ is a clique that contains the site s. For a particular clique type with n nodes any one of the nodes may represent the site s. Therefore, there exists n unique local cliques of that type.

V. Discussion and conclusion

The clique tree method extracts all the local cliques from any MRF neighbourhood system.

The clique tree method only extracts cliques from MRF neighbourhood systems because the clique tree is formed on the premise that $s \in \mathcal{G}_r \Leftrightarrow r \in \mathcal{G}_s$. For a MRF neighbourhood system defined on a homogeneous field, each neighbourhood has to be identical and symmetrical in shape.

The clique tree has been structured so that the local cliques of a particular size reside at the one level. Level 1 holds the single site clique $\{s\}$. The next level, level 2, holds all the pairwise cliques. This ordering of the cliques continues up the levels of the tree until all the local cliques have been accounted for. The tree structure makes it very easy to identify how many local cliques of a certain size exist, it is just the number of sites at the corresponding level of the tree. Therefore, the neighbourhood system of Fig. 1(b) has 1 single site clique, 8 pairwise cliques, 12 third order cliques, and 4 fourth order cliques. This is shown in Fig. 3. The total number of cliques in the local clique set is, of course, the total number of sites shown on the tree.

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